

LAST TIME: Cross Product

9/3/21

Ex.  $\vec{u} = \langle 5, 3, -1 \rangle$ ,  $\vec{v} = \langle 8, 4, 2 \rangle$  \*show all steps!

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 3 & -1 \\ 8 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} \hat{i} - \begin{vmatrix} 5 & -1 \\ 8 & 2 \end{vmatrix} \hat{j} + \begin{vmatrix} 5 & 3 \\ 8 & 4 \end{vmatrix} \hat{k}$$

$$= (3 \cdot 2 - (-1) \cdot 4) \hat{i} - (5 \cdot 2 - (-1) \cdot 8) \hat{j} + (5 \cdot 4 - 3 \cdot 8) \hat{k}$$

$$= (6 + 4) \hat{i} - (10 + 8) \hat{j} + (20 - 24) \hat{k} = 10\hat{i} - 18\hat{j} - 4\hat{k} = \boxed{\langle 10, -18, -4 \rangle}$$

↳ Recall: Prop (Algebraic Properties of cross product)

Let  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$  and  $c \in \mathbb{R}$

$$\textcircled{1} \quad \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$\textcircled{2} \quad (c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$

$$\textcircled{3} \quad \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$\textcircled{4} \quad (\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$\textcircled{5} \quad \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$$

$$\textcircled{6} \quad \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})(\vec{v} - (\vec{u} \cdot \vec{v})\vec{w})$$

\textcircled{7}  $\vec{u}$  and  $\vec{v}$  are both orthogonal to  $\vec{u} \times \vec{v}$

\textcircled{8}  $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$  with  $\theta$  the angle between  $\vec{u}, \vec{v}$

\textcircled{9}  $\vec{u} \times \vec{v} = 0$  iff  $\vec{u}$  and  $\vec{v}$  are parallel.

} geometric properties

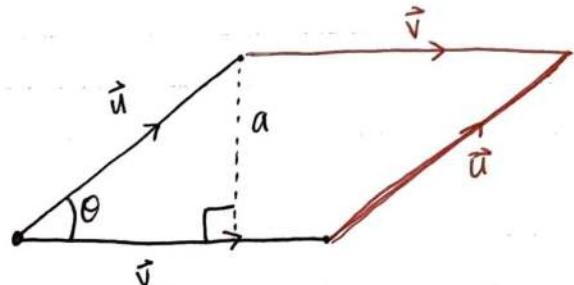
Ex. Take  $\vec{v} \times \vec{u}$  for the given vectors.

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$

$$= -\langle 10, -18, -4 \rangle$$

$$= \langle -10, 18, 4 \rangle$$

↳  $\vec{u} \times \vec{v}$  is computed w "right hand rule"



$$\sin(\theta) = \frac{a}{|\vec{v}|} \quad \therefore a = |\vec{v}| \sin(\theta)$$

∴ area of parallelogram

determined by  $\vec{u} \in \vec{v}$  is

$$A = (\text{base})(\text{height}) = |\vec{u}| a = |\vec{u}| |\vec{v}| \sin \theta$$

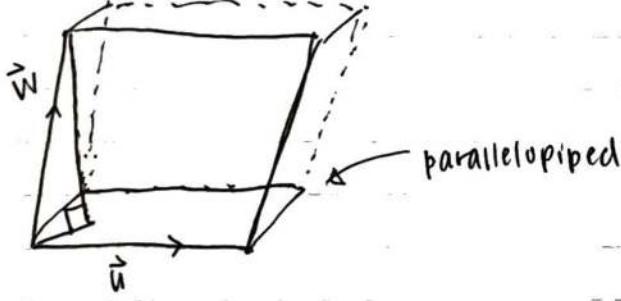
Proof of prop 8: We use the algebraic properties to compute  $|\vec{u} \times \vec{v}|^2$

$$\begin{aligned}&= |\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) \quad (\text{Properties of dot product}) \\&= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) \quad (\text{Part 5 of properties of cross}) \\&= \vec{u} \cdot ((\vec{v} \cdot \vec{v}) \vec{u} - (\vec{v} \cdot \vec{u}) \vec{v}) \quad (\text{Part 6 of properties of cross}) \\&= \vec{u} \cdot (\vec{v} \cdot \vec{v}) \vec{u} - \vec{u} \cdot (\vec{v} \cdot \vec{u}) \vec{v} \quad (\text{Properties of dot}) \\&= (\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{v}) \quad (\text{Properties of dot}) \\&= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2 \quad (\text{Properties of dot}) \\&= (|\vec{v}| |\vec{u}|)^2 - (|\vec{u}| |\vec{v}| \cos(\theta))^2 \quad \leftarrow \text{geometric interpretation of dot product} \\&= (|\vec{u}| |\vec{v}|)^2 - (|\vec{u}| |\vec{v}|)^2 \cos^2(\theta) \\&= (|\vec{u}| |\vec{v}|)^2 (1 - \cos^2(\theta)) \\&= (|\vec{u}| |\vec{v}|)^2 \sin^2(\theta) \\&= (|\vec{u}| |\vec{v}| \sin(\theta))^2 \\&\therefore |\vec{u} \times \vec{v}|^2 = (|\vec{u}| |\vec{v}| \sin(\theta))^2\end{aligned}$$

Recall:  $\theta$  is the angle between vectors  $\vec{u}$  and  $\vec{v}$ , so  $\theta \in [0, \pi]$ , so  $\sin(\theta) \geq 0$   
Hence  $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin(\theta)$

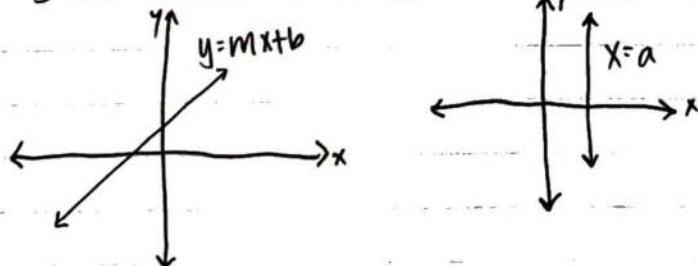
Cor: The magnitude  $|\vec{u} \times \vec{v}|$  is the area of the parallelogram determined by  $\vec{u}$  &  $\vec{v}$

prop: The scalar triple product  $\vec{u} \cdot (\vec{v} \times \vec{w})$  is the signed volume of the parallelipiped by  $\vec{u}, \vec{v}, \& \vec{w}$



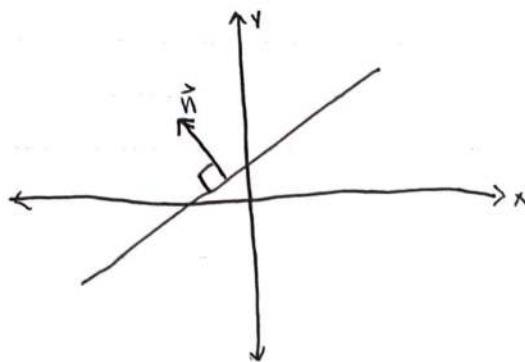
Proof is on website

### § 12.5: Lines & Planes



$$ax + by - c = 0$$

IN 2-space:



line is the set of points w/  $\vec{n} \cdot \vec{x} = c$

- Generalize this equation to 3-space.

$$\vec{n} \cdot \vec{x} = d$$

$$\text{i.e. } \langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$$

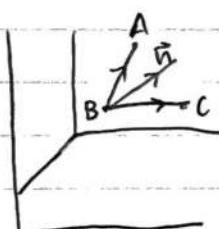
$$\text{i.e. } ax + by + cz = d$$

→ This is a plane in 3-space

NB: If we know two nonparallel vectors  $\vec{u}$  &  $\vec{v}$  which lie in the plane (i.e. their head & tail can be expressed on the plane at the same time) then  $\vec{n} = \vec{u} \times \vec{v}$  is a normal vector to the plane. i.e. it's perpendicular to every vector in the plane

ex: Find the vector equation of the plane through the points  $(0, 1, 3)$ ,  $(4, 9, 7)$ , and  $(1, 2, 3)$

solution:



$$\text{Note that the vectors } \vec{u} = \langle 4-0, 9-1, 7-3 \rangle \\ = \langle 4, 8, 4 \rangle$$

$$\text{and } \vec{v} = \langle 1-0, 2-1, 3-3 \rangle \\ = \langle 1, 1, 0 \rangle$$

lie in the desired plane

∴ we may use normal vector  $\vec{n} = \vec{u} \times \vec{v}$

$$\therefore \vec{n} = \vec{u} \times \vec{v} \quad \left| \begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 4 & 8 & 4 \\ 1 & 1 & 0 \end{array} \right|$$

$$= \langle -4, -(-4), -4 \rangle = \langle 1, -1, 1 \rangle$$

∴ the plane has equation

$$\vec{n} \cdot \vec{x} = d \text{ for some constant } d$$

$$\text{i.e. } \langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = d$$

$$\text{i.e. } x - y + z = d$$

↳ To compute  $d$ , use  $(0, 1, 3)$

$$d = 0 - 1 + 3 = 2, \text{ so the plane has equation } x - y + z = 2$$